1. Harvard Law School courses often have assigned seating to facilitate the “Socratic method.” Suppose that there are 100 first year Harvard Law students, and each takes two courses: Torts and Contracts. Both are held in the same lecture hall (which has 100 seats), and the seating is uniformly random and independent for the two courses.
2. Find the probability that no one has the same seat for both courses (exactly; you should leave your answer as a sum).   
   (b) Find a simple but accurate approximation to the probability that no one has the same seat for both courses.   
   (c) Find a simple but accurate approximation to the probability that at least two students have the same seat for both courses.

Answer :

Let's break down the problem step by step.

**(a) Exact Probability That No One Has the Same Seat for Both Courses**

To solve this, we need to find the probability that each of the 100 students has a different seat in Torts than they do in Contracts.

This is a classic example of the **derangement problem**, where we want to calculate the number of permutations of a set where no element appears in its original position. The number of such derangements DnD\_nDn​ of a set of nnn elements is given by:

Dn=n!∑k=0n(−1)kk!D\_n = n! \sum\_{k=0}^{n} \frac{(-1)^k}{k!}Dn​=n!k=0∑n​k!(−1)k​

For this specific problem, we need to calculate the probability PPP that no student has the same seat for both courses:

P=D100100!=∑k=0100(−1)kk!P = \frac{D\_{100}}{100!} = \sum\_{k=0}^{100} \frac{(-1)^k}{k!}P=100!D100​​=k=0∑100​k!(−1)k​

So the exact probability is:

P=∑k=0100(−1)kk!P = \sum\_{k=0}^{100} \frac{(-1)^k}{k!}P=k=0∑100​k!(−1)k​

**(b) Approximation to the Probability That No One Has the Same Seat for Both Courses**

For large nnn, the probability of no fixed points (no student having the same seat in both courses) approaches 1e\frac{1}{e}e1​, where eee is the base of the natural logarithm. This approximation is given by:

P≈1eP \approx \frac{1}{e}P≈e1​

This is because:

P=∑k=0n(−1)kk!≈1e for large nP = \sum\_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{1}{e} \text{ for large } nP=k=0∑n​k!(−1)k​≈e1​ for large n

**(c) Approximation to the Probability That at Least Two Students Have the Same Seat for Both Courses**

The probability that at least two students have the same seat for both courses is the complement of the probability that no student has the same seat in both courses.

If Pno same seatP\_{\text{no same seat}}Pno same seat​ is the probability that no student has the same seat for both courses, then the probability that at least two students have the same seat is:

Pat least two=1−Pno same seat≈1−1eP\_{\text{at least two}} = 1 - P\_{\text{no same seat}} \approx 1 - \frac{1}{e}Pat least two​=1−Pno same seat​≈1−e1​

So the approximate probability that at least two students have the same seat in both courses is:

Pat least two≈1−1e≈0.632P\_{\text{at least two}} \approx 1 - \frac{1}{e} \approx 0.632Pat least two​≈1−e1​≈0.632

**Summary:**

* **(a)** The exact probability that no one has the same seat for both courses is ∑k=0100(−1)kk!\sum\_{k=0}^{100} \frac{(-1)^k}{k!}∑k=0100​k!(−1)k​.
* **(b)** The approximate probability that no one has the same seat for both courses is 1e≈0.368\frac{1}{e} \approx 0.368e1​≈0.368.
* **(c)** The approximate probability that at least two students have the same seat for both courses is 1−1e≈0.6321 - \frac{1}{e} \approx 0.6321−e1​≈0.632.

1. There are 100 passengers lined up to board an airplane with 100 seats (with each seat assigned to one of the passengers). The first passenger in line crazily decides to sit in a randomly chosen seat (with all seats equally likely). Each subsequent passenger takes his or her assigned seat if available, and otherwise sits in a random available seat. What is the probability that the last passenger in line gets to sit in his or her assigned seat?

Answer :

This is a well-known problem in probability, often referred to as the "airplane seat problem."

**Problem Recap:**

* There are 100 passengers and 100 seats.
* The first passenger chooses a seat at random.
* Each subsequent passenger sits in their assigned seat if it's available; otherwise, they choose a seat at random from those that remain.
* We need to find the probability that the last (100th) passenger sits in their assigned seat.

**Solution:**

Interestingly, the probability that the last passenger ends up in their assigned seat turns out to be independent of the number of passengers (as long as there are at least 2 passengers).

**Explanation:**

* For the 100th passenger to end up in their assigned seat, all preceding passengers must not take that seat.
* The process is recursive, and it turns out that the probability of the 100th passenger sitting in their own seat is the same regardless of whether the first passenger took their own seat, the 100th seat, or any other seat.

This situation simplifies to the following:

* If the first passenger sits in their own seat, the probability that the 100th passenger sits in their own seat is 1.
* If the first passenger sits in the 100th passenger’s seat, then the 100th passenger cannot sit in their assigned seat (probability is 0).
* For any other seat that the first passenger takes, the problem effectively reduces to the same scenario but with one fewer passenger and one fewer seat. The key insight is that the probability remains constant across all such scenarios.

This leads to the conclusion that the probability the last passenger sits in their own seat is 12\frac{1}{2}21​.

**Final Answer:**

The probability that the last (100th) passenger gets to sit in their assigned seat is **12\frac{1}{2}21​** or **50%**.